Transition from Abelian to non-Abelian quantum liquids in the second Landau level

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The search for non-Abelian quantum Hall states in the second Landau level is narrowed to the range of filling factors \( 7/3 < \nu_c < 8/3 \). In this range, the analysis of energy spectra and correlation functions, calculated including finite width and Landau-level mixing, supports the prominent non-Abelian candidates at \( \nu_c = 5/2 \) (Moore-Read) and 12/5 (Read-Rezayi). Outside of it, the four-flux noninteracting composite fermion model is validated. The borderline \( \nu_c = 7/3 \) state is adiabatically connected to the Laughlin liquid, but its short-range correlations and charge excitations are different.

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The incompressible quantum liquids (IQLs) (Ref. 1) formed in a high magnetic field \( B \) by two-dimensional electrons filling various fractions \( \nu \) of different Landau levels (LLs, \( n = 0, 1, \ldots \)) have been the subject of extensive studies ever since the famous discovery of the fractional quantum Hall (FQH) effect.\(^{2}\) The most recent storm of interest is motivated by the concept of “topological quantum computation”\(^{3,4}\) employing non-Abelian statistics of the Moore-Read “pfaffian” wave function\(^{5,6}\) believed to describe the FQH state in a half-filled LL\(_1\). Other wave functions with different complexities of braiding statistics have also been proposed,\(^{7,8}\) elevating the convincing demonstration of non-Abelian statistics in a real physical system to the challenge of greatest current importance.

With the partially filled LL\(_0\) successfully described by the composite fermion (CF) theory\(^{10}\) and with ordered electron phases favored in higher LLs, signatures of non-Abelian statistics are most strongly anticipated in the few known IQLs in LL\(_1\). Crucial recent experiments in this LL include confirmation\(^{11}\) of the quasiparticle (QP) charge of \( 1/2 \) for the half-filled state at \( \nu_c = 5/2 \) and careful measurements\(^{12-14}\) of the minute excitation gaps. In theory, the most recent advances have been related to the breaking of particle-hole symmetry in the Moore-Read state\(^{15}\), the role of layer width\(^{16}\) and LL mixing\(^{17,18}\) in real systems, and the nature of QPs\(^{19,20}\).

Despite intensive studies, connection of the FQH states in LL\(_1\) (\( \nu_c = \frac{5}{2}, \frac{7}{3}, \frac{12}{5}, \frac{11}{5} \)) to the few proposed wave functions is not conclusively established, and in some cases it is only tentatively assumed for the lack of other candidates. This is an urgent problem as the connection of some of these wave functions to the particular conformal field theories is precisely what fuels the anticipation of non-Abelian statistics in nature. Another unresolved puzzle is the discrepancy\(^{14}\) between the experimental and numerical gaps at \( \nu_c = \frac{5}{2} \), which undermines the understanding of this state in simple models assuming spin polarization or decoupling from the crystal lattice. On the other hand, a wealth of IQLs found in various systems (electrons or CFs at different fillings of different LLs, in layers of varied width \( w \)) invites a more general question of possible IQLs with arbitrary interactions \( V \).

In this Rapid Communication we demonstrate (numerically) that non-Abelian IQLs in LL\(_1\) can only emerge in the narrow particle-hole symmetric range of filling factors \( \frac{7}{3} < \nu_c < \frac{8}{3} \). In this range, the known non-Abelian candidates at \( \nu_c = \frac{5}{2} \) and \( \frac{12}{5} \) are closely examined (including finite layer width and LL mixing) and found to have favorable correlation energies. Outside of it, the Abelian ground states of non-interacting CFs carrying four flux quanta repeat in both lowest LLs. The borderline \( \nu_c = \frac{7}{3} \) ground state is adiabatically connected to the Laughlin state, but it has a smaller gap, distinct short-range correlations, negative quasihole (QH) energy in narrow layers, and probably different quasielectrons (QEs).

We study spin-polarized \( N \)-particle systems on a unit sphere with the magnetic monopole of strength \( 2Q/(hc/e) \) inside.\(^{21}\) In this geometry, LL\(_m\) is a shell of angular momentum \( \ell = Q+m \), and different \( N \)-body wave functions at the same \( \nu \) are distinguished by a “shift” \( \gamma \) between LL degeneracy and \( \nu^{-1}N \). In contrast to previous exact diagonalization calculations, we do not confine ourselves to the systems with particular \( V \) but search for the universality classes \((\nu, \gamma)\) of gapped ground states with arbitrary interactions, attainable in realistic systems (electrons or CFs) with the suitable choice of \( n \) and \( w \).

We begin by recalling that many-body dynamics in a degenerate LL is determined by an interaction pseudopotential, defined\(^{21}\) as the dependence of the pair energy \( V \) on the relative angular momentum \( m=1,3, \ldots \). Moreover, \( V_m \) induces particular correlations through its deviation from a straight line over any consecutive \( m \)'s.\(^{22}\) Hence, it is known that low-energy spectra of \( V_m \) are reproduced by a suitable effective pseudopotential \( U_m \) with only a few (positive or negative) coefficients.

We have used \( U = [U_1, U_3, U_5] \) in the search for IQLs with (nearly) arbitrary interactions. Higher terms (causing long-range order) were ignored. On the other hand, inclusion of \( U_5 \) was needed for an accurate description of the two lowest (electron or CF) LLs known to host IQL states, while it still allowed for useful graphical representation of the ground-state properties in an effectively two-dimensional space of (normalized) parameters \( U_\nu \).

We looked at various finite systems \((N,2\ell)\) in search of the series of gapped ground states at \( 2\ell\nu = \nu^{-1}N - \gamma \). A few maps of the “neutral” gap \( \Delta \) (gap to the first-excited state in the same spectrum) appear in Fig. 1. The IQL candidates are the maxima in \( \Delta \) repeating regularly for different \( N \). Their
Quite definitively the earlier expectations that the Moore-Read state in other half-filled LLs coincides with a peak in similar maps and shifts $\gamma$ assigned to each finite system $N=1,3,5$, marked in each corner. Points relevant for the electrons in LL $1$, whose actual Coulomb pseudopotentials in different LLs, whose approximate positions (A)-(D) are indicated in frame (i).

Let us begin with a half-filled LL represented by $2\ell=2N-3$. The maximum near $U_{(B)}=[\frac{5}{2},\frac{1}{2},0]$ found for each $N$ is relevant for the electrons in LL $1$, whose actual Coulomb pseudopotential is almost linear between $m=1$ and $5$. The maximum coincides with a peak in similar maps (not shown) of the squared overlap of the ground state of $U$ with the exact Moore-Read state and also with an essentially zero-level minimum of the triplet Haldane amplitude. This confirms quite definitively the earlier expectations that the Moore-Read ground state indeed occurs for a class of pseudopotentials close to that of LL $1$ and that its accuracy depends sensitively on the fine-tuning of the leading $V_m$'s, achieved by adjusting the layer width $w$. Remarkably, it precludes the Moore-Read state in other half-filled LLs (e.g., in the second CF LL, called LL $1^*$, with dominant repulsion at $m=3$), where the observed FQH state must be different).

In similar maps for $\nu=\frac{1}{3}$, significant gaps are found in different areas for the series of states with $\gamma=3$ (Laughlin) and 7. The latter was proposed earlier for both LL $1$ (in Ref. 22) and for LL $1^*$ (in Ref. 26) but, despite numerical evidence for pairing, its wave function remains unknown. It is clear from Figs. 1(c) and 1(d) that in the vicinity of $U_{(B)}$, relevant for LL $1$, competition between the $\gamma=3$ and 7 states must be resolved more carefully.

For $\nu=\frac{2}{5}$, Jain's series with $\gamma=4$ correctly represents the ground state in LL $0$. In LL $1$, this series competes with two others: the parafermion $\gamma=-2$ state and a recently proposed state with $\gamma=2$. Especially for the latter, Fig. 1(e) appears suggestive of a gap emerging around $U_{(B)}$. The subsequent careful analysis of the competition between these three universality classes is crucial because the candidate states with $\gamma=\pm 2$ are both non-Abelian, in contrast to the $\gamma=4$ Jain state. Finally, at $\nu=\frac{1}{5}$, any positive $U=[U_1, U_3,0]$ yields an exact Laughlin state. Figure 1(h) shows that it is true description of the FQH states in both LL $0$ and LL $1$. On the other hand, its relevance to the FQH effect observed in LL $1^*$ is doubtful (indeed, an alternative series with $\gamma=9$ appears to have more favorable correlations).

Guided by the maps of $\Delta$, pair/triplet amplitudes, and various overlaps we now focus on the FQH states in LL $1$. In Fig. 2 we seek confirmation of the IQL candidates in cusps of the dependence of the ground-state energy per particle $E$ on the LL degeneracy. The cusps are most pronounced in the plots of $\delta E_2=E_{2l+1}-E_{2l}$. For an IQL, its (positive) value gives the QP gap $\Delta$ times the number of QPs created per flux quantum.

Besides complementing the maps of Fig. 1 in the identification of the universality classes of particular IQLs, Fig. 2 reveals a simple connection between statistics and filling factor in LL $1$. At $\nu<\frac{1}{3}$ the same Laughlin/Jain IQLs occur in LL $1$ and LL $0$, with high overlaps and similar gaps. This similarity, earlier pointed out in Ref. 27, is caused by sufficiently high $V_1$ and similar $V_{m=3}$ in both LLs. It validates the non-
interacting CF model\textsuperscript{10} with four-flux quanta attached to each electron in LL\textsubscript{1}.

In contrast, at \(\nu > \frac{1}{2}\) the Jain sequence of LL\textsubscript{0} is replaced in LL\textsubscript{1} by a different set of IQLS, including the pfaffian and antipfaffian pair at \(\nu = \frac{1}{2}\) and, apparently, a different \(\nu = \frac{5}{2}\) state with \(\gamma = 2\) (Fig. 2 shows no signatures of the parafermion \(\nu = \frac{5}{2}\) series\textsuperscript{7} with \(\gamma = -2\)). The breakdown of the noninteracting two-flux CF model in LL\textsubscript{1} opens, exclusively at \(\frac{1}{2} < \nu < \frac{3}{2}\), a possibility for other IQLS, including several suggested more exotic states with various non-Abelian statistics of their QPs.

For the borderline \(\nu = \frac{1}{2}\) state, separating the Abelian from (possibly) non-Abelian states, our calculation reinforces the theory of Ref. 27. This state has a moderate overlap with the Laughlin ground state of LL\textsubscript{0}, despite falling into the same theory of Ref. 27. This state has a moderate overlap with the Laughlin/Jain states in LL\textsubscript{0} or LL\textsubscript{1}.

Let us now examine the crossover \(\nu = \frac{1}{2}\) ground states in LL\textsubscript{1}. They have only moderate squared overlaps with the Laughlin states: 0.292 (0.501), 0.253 (0.510), and 0.333 (0.549) for \(N=10\), 12, and 14, when \(w=0(3\lambda)\). This reflects quite different correlations (e.g., for \(N=12\) and \(w=3\lambda\), the pair amplitudes at \(m=1, 3, 5, 7\) and \(w=0, 1/2, 0.13, 0.17, 0.10\) in LL\textsubscript{1} as opposed to 0.00, 0.18, 0.13, and 0.09 in LL\textsubscript{0}). The magnetoroton band is also absent in the spectra of LL\textsubscript{1}. The low-energy states resembling Laughlin QEs and QHs are found at \(2\xi = 3N - 3 = 1\), but they are not generally the lowest states in their spectra.

Although we have checked that the eigenstates adiabatically connected to the Laughlin QPs become the lowest states under bias, their joint energy \(\Delta = E_{QH} + E_{QE}\) is unreasonably high. Figures 4(a) and 4(b) show the size extrapolation of QE and QH energies \(e\), and width dependences of the extrapolated \(e\)’s. Remarkably, \(E_{QH} > 0\) requires finite width \(w > 3\lambda\). Negative QH energies in narrow layers are also found in other states in LL\textsubscript{1} identified in Fig. 2(b). For instance, \(e = \frac{1}{2} \left( E_{2\xi=1} - E_{2\xi} \right)\) plus the size corrections\textsuperscript{29} are plotted in Figs. 4(c) and 4(d) for \(\nu = \frac{1}{2}\).

Note that the gap \(\Delta \) at \(\nu = \frac{3}{2}\) is not immediately weakened in wider wells (in contrast to Ref. 29 where only data for \(N=10\) and 14 were used for \(w > 0\)). The initial gap enhancement in wider wells is even more pronounced with the inclusion of LL mixing (here, the corrections \(\Delta E\) were calculated for \(N=14\) at \(2\xi = 2N - 3\) and \(2N - 3 \pm 1\)). While the reasonable match between the experimental gaps and \(\Delta\) (rather than
However, a larger difference between lowest Landau level are added for comparison. In N
eq 2, remains a mystery, the difference between the latter two is well understood as the QE-QH attraction.\(^{29}\) However, a larger difference between Δ and ~Δ at ν = 5/3 suggests that Laughlin QEs may not be the elementary negative carriers in LL\(_1\).

In conclusion, we have found the range of filling factors \(\frac{1}{3} < ν < \frac{2}{3}\) in LL\(_1\) in which the emergence of non-Abelian statistics is possible. Inside this range, our calculations including finite layer width and LL mixing demonstrate that the spin-polarized ground states at ν = 1/2 and 2/3 are described by the "pfaffian" and "parafermion" wave functions (both non-Abelian). Outside, the Jain states of noninteracting CFs repeat in both lowest LLs, precluding more exotic phases. The borderline ν = 5/3 state is adiabatically connected to the Laughlin liquid but has a smaller gap and distinct excitations.

Recently, it has come to our attention that the connection of the ν = 5/3 ground state with the Moore-Read wave function was also concluded in Ref. 30, from a map equivalent to our Fig. 1(b).

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\[\Delta\] at ν = 3/2