Global Phase Diagram of the Fractional Quantum Hall Effect Arising from Repulsive Three-Body Interactions

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The model of fermions in a magnetic field interacting via a purely three-body repulsive interaction has attracted interest because it produces, in the limit of short range interaction, the Pfaffian state with non-Abelian excitations. We show that this is part of a rich phase diagram containing a host of fractional quantum Hall states, a composite fermion Fermi sea, and a pairing transition. This is entirely unexpected, because the appearance of composite fermions and fractional quantum Hall effect is ordinarily thought to be a result of strong two-body repulsion. Recent breakthroughs in ultracold atoms have facilitated the realization of such a system, where this physics can be tested.

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the lowest energy neutral excitations [15,16]; and their Fermi sea describes the compressible state at the half filled LL [17,18]. In certain cases, the residual interaction between the CFs plays an important role. Certain fractions such as \( \nu = 4/11 \) represent a FQHE of CFs [19,20]. At \( \nu = 5/2 \), a residual attraction between CFs [21] is believed to cause their \( p_x \pm ip_y \) pairing, described by the above mentioned Pfaffian wave function [2–5].

We consider fully spin polarized fermions in the lowest LL, subject to a model three-body interaction in which all but the first two relevant pseudopotentials, \( V^{(3)}_3 = A \) and \( V^{(3)}_5 = B \), are set to zero. (For three-body interaction, \( m = 1 \) and 2 are excluded by the Pauli principle and \( m = 4 \) by symmetry.) The most reliable method available for determining the states produced by a general interaction is that of exact diagonalization, for which we employ a geometry [7] that has \( N \) fermions moving on the surface of a sphere, subjected to a total radial magnetic flux of \( 2Q \) (in units of \( hc/e \)). The projected Hamiltonian assumes the form [22]

\[
\hat{H}_{\text{three-body}} = A \sum_{i<j<k} P_{ijk}(3Q - 3) + B \sum_{i<j<k} P_{ijk}(3Q - 5),
\]

where \( P_{ijk}(L) \) projects the state of the three particles \((i, j, k)\) into the subspace of total orbital angular momentum \( L \). This Hamiltonian admits an exact solution for the ground state at \( \nu = 1/2 \) (the Pfaffian) when \( B = 0 \), and for the ground state at \( \nu = 2/5 \) when both \( A \) and \( B \) are nonzero [23]. We consider below the region \( \nu > 2/5 \); there is no FQHE for this model for \( \nu < 2/5 \). Increasing \( B/A \) amounts to extending the range of the interaction.

We begin by testing the evolution of the Pfaffian solution as a function of the range of the interaction. The overlap of the exact ground state of \( \hat{H}_{\text{three-body}} \) with the Pfaffian wave function is seen in Fig. 1(a) to decay rapidly with increasing \( B/A \), indicating a transition into some other state. To gain insight into the nature of the new state we monitor the low-energy spectrum, which reveals a fundamental reorganization of the low-energy spectrum of \( H_{\text{three-body}} \) for 14 fermions at \( 2Q = 25 \). Panels (c) and (d) display comparison of the exact ground states of \( H_{\text{three-body}} \) at \( \nu = 2/3, 3/5, 4/9, 3/7, 4/5, 7/9, \) and \( 5/7 \) with the corresponding wave functions of IQHE states of composite fermions (for \( N = 22, 18, 16, 15, 32, 30 \) and 24, respectively).

![FIG. 1](color online). Evolution of various states as a function of \( B/A \), i.e., the range of the three-body interaction. Panel (a) shows the squared overlap of the lowest energy \( L = 0 \) eigenfunction of \( H_{\text{three-body}} \) (which is also the ground state for \( B/A < 0.3 \)) with the Pfaffian wave function as well as with the \( L = 0 \) state of weakly interacting composite fermions at \( \nu = 1/2 \) (\( N = 18, 2Q = 33 \)). The spherical geometry is used for these calculations, and \( L \) is the total orbital angular momentum. Panel (b) shows the squared overlaps of all CF states in the lowest energy band with the corresponding lowest energy eigenstates of \( H_{\text{three-body}} \) for 14 fermions at \( 2Q = 25 \). Panels (c) and (d) display comparison of the exact ground states of \( H_{\text{three-body}} \) at \( \nu = 2/3, 3/5, 4/9, 3/7, 4/5, 7/9, \) and \( 5/7 \) with the corresponding wave functions of IQHE states of composite fermions (for \( N = 22, 18, 16, 15, 32, 30 \) and 24, respectively).

as also evident by an almost exact agreement between their Coulomb energies (Fig. 2, lowest panels). Figures 1(a) and 1(b) demonstrate that as the overlap with the Pfaffian wave function drops, the overlap with the weakly interacting CF wave functions rapidly grows, approaching a high maximum at approximately \( B/A = 0.5 \). These results establish a phase transition from the paired CF state into the CF Fermi sea as a function of the range of the three-body interaction, which appears continuous to the extent we can surmise from our finite size study. We note that we have shown here (and below) results only for the largest system that we have studied for each filling factor, but the smaller systems are fully consistent with our conclusions.

The formation of CFs at \( \nu = 1/2 \) suggests the possibility of another phenomenon associated with CFs. We first study the states of \( H_{\text{three-body}} \) at several fractions of the form \( \nu = n^*/(2n^* \pm 1) \) and \( \nu = 1 - n^*/(2n^* \pm 1) \). Figure 1(c) depicts the ground state overlap as a function of \( B/A \), approaching a very high maximum at what we term the “optimal” value of \( B/A \). Figure 2 again illustrates a drastic reorganization of the low-energy spectrum of \( H_{\text{three-body}} \) as \( B/A \) is turned on. It is evident that near the optimal \( B/A \), all states studied here (except \( \nu = 1/2 \)) are incompressible, in that they have a uniform \( (L = 0) \) ground state separated from excitations by a robust gap. Furthermore, the spectrum bears a striking resemblance to the weakly interacting CF spectrum, shown in the bottom panel, not only for the \( L = 0 \) ground state but also for the low-energy branch of
neutral excitations. The overlaps confirm the ground state as the CF-IQHE state and the low-energy excitations as the CF excitons.

Figures 1(d) and 3 demonstrate that the three-body interaction not only creates CFs carrying two vortices ($^2$CFs) in the region $2/3 \geq \nu > 2/5$ but also CFs carrying four vortices ($^4$CFs) for $\nu > 2/3$ at fillings of the form $\nu = 1 - n^*/(4n^* + 1)$. In fact, $^4$CFs are more robust; for $\nu < 2/3$ it requires a nonzero $B/A$ to create weakly interacting CFs, whereas for $\nu \geq 2/3$ the CF physics is established already at $B/A = 0$.

We have also studied numerous systems in between the special filling factors shown above, and in all cases we find that for a range of $B/A$ values the physics is consistent with weakly interacting CFs. Specifically, the low-energy band of states is well described, qualitatively and quantitatively, in terms of CF quasiparticles or CF quasiholes on top of a CF-IQHE state.

The three-body interaction does not respect particle-hole symmetry in the lowest LL, which is responsible for the qualitatively distinct physics for $\nu < 2/5$ and $3/5 < \nu < 1$, and also for the Pfaffian state, which is not particle-hole symmetric. A surprising outcome of this work is that for a range of $B/A$ values, the three-body interaction behaves similarly as the two-body Coulomb interaction insofar as the low-energy physics of the correlated states is concerned, implying a partial restoration of the particle-hole symmetry in the region $3/5 > \nu > 2/5$ when the longer range part of the three-body interaction is turned on.

Our principal conclusion is summarized in the phase diagram in Fig. 4. The emergence of CFs and the FQHE for three-body interactions is unexpected. The canonical model for the FQHE is that of pairwise interaction.
a strong short range repulsion, and CFs are thought to materialize to minimize the short range part of the two-body interaction. This is most evident from the fact that the “unprojected” wave functions of CFs [8] vanish much faster than what is required by the Pauli principle when two particles come close. Nevertheless, the appearance of almost free CFs enables an understanding of the physics of lowest LL fermions with three-body interaction at a level that is almost as detailed and accurate as that available for the Coulomb interaction.

Finally, although our study deals with fermions, experience from earlier work [25,26] suggests that for appropriate three-body interactions bosons will also composite fermionize by capturing a single quantized vortex (not to be confused with the vortex in the order parameter field of the Bose-Einstein condensate), to produce the FQHE at \( \nu = n^*/(n^* + 1) \) and a CF Fermi sea or a paired state at \( \nu = 1 \). We have not yet investigated this possibility.

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**FIG. 4 (color online).** The phase diagram of the FQHE for three-body interaction. The region where the physics is described in terms of weakly interacting composite fermions is shaded in blue (dark gray), and the region of paired composite fermions in yellow (light gray). At \( \nu = 1/2 \) the Pfaffian FQHE state undergoes a transition into the CF Fermi sea with increasing \( B/A \). Each vertical line indicates the range where squared overlap of the exact ground state of the three-body interaction with the weakly interacting CF state exceeds 90% (for our largest system, see Fig. 1), with the dot marking the position of the maximum overlap. The approximate phase boundaries (roughly connecting numerical data points) are to be viewed as being semi-quantitative.